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Unsteady and nonlinear effects near the cusp lines of the Kelvin ship-wave pattern

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According to the linearized water-wave theory, a localized pressure source travelling at constant speed on the surface of deep water generates the classical Kelvin ship-wave pattern, which follows behind the source and is confined within a sector of half-angle equal to 19.5°. In this paper, an asymptotic theory is developed which takes into account finite-amplitude and unsteady effects near the boundaries of the Kelvin sector, the so-called cusp lines, where the far-field wave disturbance takes the form of a modulated wavepacket. A nonlinear equation governing the spatial and temporal evolution of the wavepacket envelope is derived. It is shown that, for a pressure source turned on impulsively, a nonlinear steady state is reached. All unsteady effects are found in a region of finite extent which moves away from the source. Numerical calculations indicate that the steady-state nonlinear response is very similar to the steady-state linear response.

1. Introduction

The wave pattern generated by a ship travelling at constant speed on the surface of deep water was first discussed by Kelvin (1905) on the basis of linearized water-wave theory. He modelled the ship as a prescribed localized pressure distribution and calculated the far-field, steady-state wave disturbance using the method of stationary phase. His analysis shows that the wave pattern consists of both transverse and diverging waves which are found behind the source and are confined within a sector of half-angle equal to 19.5°. Near the boundaries of the Kelvin sector, the so-called cusp lines, where the transverse and diverging waves meet, there is a finite transition region in which the disturbance switches from oscillatory to exponentially decaying. These transition regions near the cusp lines, as well as the precise behaviour close to the track of the source, were analysed in detail by Ursell (1960).

Kelvin's solution, based entirely on the linearized water-wave theory, does not take into account any nonlinear effects due to the exact nonlinear free-surface boundary conditions. Furthermore, it does not incorporate the appropriate boundary conditions at the ship hull, so that it is not expected to model accurately the flow field near the ship. These two issues have attracted a great deal of attention in recent years, primarily because they are important in calculating ship-wave resistance.

A method to include weakly nonlinear effects in the Kelvin ship-wave pattern, which seems to be analytically tractable, is to use a perturbation expansion in terms of a wave-amplitude parameter: Kelvin's solution provides the leading-order approximation and higher-order nonlinear corrections are found by solving a hierarchy of linear, inhomogeneous problems. Along these lines, Gadd (1969) proposed a perturbative scheme which, in principle, takes into account both the boundary conditions on the ship surface and the free-surface nonlinearities. However, in a published discussion following Gadd's paper, Lighthill (see Gadd 1969) criticized this treatment of free-surface nonlinearity: apart from being very tedious to carry out, a regular perturbation expansion cannot predict any large changes in the wave disturbance, which may be generated at large distances from the source owing to weakly nonlinear effects. He also suggested that the problem could be handled by Whitham's theory (Whitham 1974) which applies to slowly varying nonlinear wavetrains.

Howe (1967) attempted to apply Whitham's theory to the ship-wave problem, but was unable to do so because the wave pattern consists of two wavetrains, transverse and diverging, which also are not slowly varying close to the ship. Instead, he applied Whitham's theory to a single nonlinear wavetrain, generated by a uniform stream past a slowly varying wall, and found that nonlinear effects lead to phase jumps. However, such phenomena were not observed in laboratory experiments (Newman 1970).

Working from a somewhat different viewpoint, Hogben (1972) examined the effect of wave interactions on the phase geometry of the Kelvin wave pattern generated by a source travelling in a channel of finite width. Including only a finite number of discrete waves, he found no appreciable nonlinear effects. However, this approach is perhaps inadequate in the case of an unbounded channel, where a continuum of modes exist, and is certainly invalid close to the cusp lines, where the transition regions described by Ursell (1960) are present.

A little earlier, Newman (1971) investigated the possibility of third-order nonlinear interactions between transverse and diverging waves and came to the conclusion that no such interactions are possible, except perhaps near the cusp lines. Invoking Lighthill's radiation condition (Whitham 1974, p. 447), he proceeded to calculate the effect of nonlinearity close to the cusp lines in terms of a regular perturbation expansion akin to that proposed by Gadd (1969). He reached the surprising conclusion that no nonlinear steady state exists, in the sense that the third-order nonlinear correction diverges logarithmically in the far field. These findings are interesting and suggest that nonlinear effects are particularly strong close to the cusp lines. However, Lighthill's criticism may still be appropriate: the divergence of the third-order nonlinear correction indicates that the regular perturbation expansion breaks down in the far field and a uniformly valid representation is needed.

The present work, which was originally motivated by the findings of Newman (1971), examines the far-field nonlinear behaviour of the Kelvin wave pattern near the cusp lines. Two questions are addressed: first, the existence of a nonlinear steady state; and, secondly, the extent to which such a steady state, if it exists and is stable, is different from the steady state predicted by linear theory. To answer these questions, the wave disturbance near the cusp lines is represented as a modulated wavepacket. Using the method of multiple scales, a nonlinear evolution equation for the wave envelope is derived, which enables us to obtain a uniformly valid representation of the far-field response. It is demonstrated that, contrary to the conclusion reached by Newman, a nonlinear steady state is approached; all unsteady effects are confined within a certain finite region which moves away from the source. Furthermore, numerical solutions of the nonlinear evolution equation indicate that the nonlinear response is very similar to the steady-state linear response.

The approach to the nonlinear steady state is investigated on the assumption that the source is turned on impulsively. This choice of initial condition is more realistic than Lighthill's radiation condition, invoked by Newman (1971), which assumes that the source is turned on very slowly (adiabatically) from zero to its steady-state form. In linear steady-wave problems, where the final steady state is independent of initial conditions, the use of Lighthill's technique is extremely convenient, for it gives the steady-state response with the minimum amount of manipulation. However, as pointed out by Akylas (1984*b*), in nonlinear problems the long-time behaviour of the response may depend crucially on the choice of initial conditions.

2. Asymptotic behaviour near the cusp lines

Consider a localized pressure source travelling with constant speed along a straight path on the surface of deep water. Alternatively, the source may be taken to be stationary in the presence of a uniform stream U. Our interest centres on the wave disturbance near the cusp lines of the Kelvin pattern, which are inclined with respect to the streamwise direction by the angles $\theta = \pm 19.5°$ ($\tan\theta = 2^{-\frac{3}{2}}$). Accordingly, it proves convenient to choose a coordinate system such that the x-axis is along the cusp line $\theta = \tan^{-1} 2^{-\frac{3}{2}}$, the y-axis is vertically upwards from the undisturbed free surface, and the z-axis is in the plane of the undisturbed free surface, at right angles to x and y.

Dimensionless (primed) variables are adopted according to

$$(x,y,z) = \frac{U^2}{g}(x',y',z'), \quad t = \frac{U}{g}t', \quad \eta = a\eta', \quad \phi = aU\phi', \quad p = ag\rho p',$$

where $y = \eta(x,z,t)$ is the free-surface elevation, $\phi(x,y,z,t)$ is the perturbation velocity potential, $p(x,z)$ is the prescribed pressure, a is a typical wave amplitude and g, ρ denote the gravitational acceleration and the water density respectively. Dropping the primes, the governing equations consist of Laplace's equation for ϕ,

$$\phi_{xx}+\phi_{yy}+\phi_{zz} = 0 \quad (-\infty < y < \epsilon\eta), \tag{1}$$

together with the kinematic and dynamic boundary conditions at the free surface $y = \epsilon\eta$

$$\eta_t + \cos\theta\eta_x - \sin\theta\eta_z - \phi_y + \epsilon(\phi_x\eta_x + \phi_z\eta_z) = 0 \quad (y = \epsilon\eta), \tag{2}$$

$$\phi_t + \eta + \cos\theta\phi_x - \sin\theta\phi_z + \tfrac{1}{2}\epsilon(\phi_x^2+\phi_y^2+\phi_z^2) = -p \quad (y = \epsilon\eta), \tag{3}$$

and the condition at infinity

$$|\nabla\phi| \to 0 \quad (y \to -\infty). \tag{4}$$

The dimensionless parameter $\epsilon = ag/U^2$, being proportional to a, is a measure of nonlinearity as well as a measure of the source strength.

To motivate the ensuing analysis and show the potential significance of nonlinear effects near the cusp lines, we first recall the wave behaviour predicted by linear theory ($\epsilon = 0$). Using an extended version of the method of stationary phase, Ursell (1960) demonstrated that near the cusp lines the linear, steady-state, far-field response takes the form

$$\eta \sim \frac{C}{x^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{3}{2^{\frac{1}{2}}}\frac{z}{x^{\frac{1}{3}}}\right)e^{i\alpha} + \text{c.c.} \quad (x \gg 1), \tag{5}$$

where

$$\alpha = \boldsymbol{k}_0\cdot\boldsymbol{x}, \tag{6a}$$

$$\boldsymbol{k}_0 = \tfrac{3}{2}(-\cos\gamma, \sin\gamma), \quad \boldsymbol{x} = (x,z), \quad \tan\gamma = 2^{\frac{1}{2}}; \tag{6b}$$

Ai is the standard Airy function, C is a certain constant related to the source strength, and c.c. stands for complex-conjugate. Equation (5) shows that, close to the cusp line

$z = 0$, the response takes the form of a modulated wavepacket with wavenumber $\boldsymbol{k}_0$; as $x \rightarrow \infty$, the wave amplitude decays like $x^{-\frac{1}{3}}$ while the extent of the wavepacket in the z-direction grows like $x^{\frac{1}{3}}$. Accordingly, in the far-field, linear dispersive effects are $O(x^{-\frac{4}{3}})$ whereas nonlinear effects are $O(\epsilon^2/x)$; dispersive effects decay faster than nonlinear effects and thus, for any finite ϵ, linearized theory becomes non-uniform at sufficiently large x: nonlinear effects cannot be neglected in comparison with linear dispersive effects for $x \geqslant O(\epsilon^{-6})$. Also, for $x = O(\epsilon^{-6})$, $\eta = O(\epsilon^2)$.

The far-field expression (5) was obtained by Ursell (1960) by deriving asymptotic expansions of the exact linear response. In discussing the nonlinear response, however, this approach is most inconvenient, as the exact nonlinear solution is not available. Instead, guided by the results of the linear theory, we assume that, in the far field ($x \gg 1$), the disturbance still takes the form of a modulated wavepacket and use asymptotic expansions directly, valid for $0 < \epsilon \ll 1$:

$$\phi \sim \epsilon^2\phi_1 + \epsilon^3\phi_2 + \epsilon^3\phi_0 + O(\epsilon^4), \quad \eta \sim \epsilon^2\eta_1 + \epsilon^3\eta_2 + \epsilon^3\eta_0 + O(\epsilon^4), \tag{7}$$

with

$$\phi_1 = A(X, Y, Z, T)\, e^{k_0 y}\, e^{i\alpha} + \text{c.c.}, \quad \eta_1 = S(X, Z, T)\, e^{i\alpha} + \text{c.c.}, \tag{8a}$$

$$\phi_2 = A_2(X, Y, Z, T)\, e^{2k_0 y}\, e^{2i\alpha} + \text{c.c.}, \quad \eta_2 = S_2(X, Z, T)\, e^{2i\alpha} + \text{c.c.}, \tag{8b}$$

$$\phi_0 = A_0(X, Y, Z, T), \quad \eta_0 = S_0(X, Z, T), \tag{8c}$$

where $k_0 = |\boldsymbol{k}_0| = \frac{3}{2}$. Here X, Y, Z, T denote the wavepacket envelope variables

$$Z = \epsilon^2 z, \quad Y = \epsilon^2 y, \quad X = \epsilon^6 x, \quad T = \epsilon^6 t. \tag{9}$$

The form of the above expansion as well as the scalings of X, Y, Z are suggested by the qualitative arguments made earlier, whereas T is chosen such that unsteady and dispersive effects balance in the far field. In the expansion (7), η_2, ϕ_2 represent the second harmonic and η_0, ϕ_0 the mean flow generated by the nonlinear interactions.

The main goal is to derive an evolution equation for S, the envelope of the free-surface elevation. This is done by substituting the expansion (7) into Laplace's equation (1) and the free-surface conditions (2), (3) and solving perturbatively for the mean flow, the second harmonic, and the primary harmonic. However, a little insight can reduce substantially the amount of algebra involved in the derivation. It is helpful to note that, in deep water, the induced mean flow does not participate in the dynamics of the primary and, thus, can be ignored, so that the derivation of the evolution equation can be carried out in two steps: first, the linear dispersive terms can be obtained from the linear water-wave dispersion relation which, in the frame of reference moving with the source, reads

$$\omega(l, m) = (l^2 + m^2)^{\frac{1}{4}} + l \cos\theta - m \sin\theta, \tag{10}$$

where ω is the frequency and l, m are the x- and z-components of the wavenumber vector $\boldsymbol{k}$. For $\boldsymbol{k} = \boldsymbol{k}_0$, referring back to (6b), it is seen that

$$l_0 = -\frac{3^{\frac{1}{2}}}{2}, \quad m_0 = \left(\frac{3}{2}\right)^{\frac{1}{2}}. \tag{11}$$

Therefore, the linear dispersive part of the evolution equation is

$$\left\{\epsilon^6 \frac{\partial}{\partial T} + i\omega\left(l_0 - i\epsilon^6 \frac{\partial}{\partial X}, m_0 - i\epsilon^2 \frac{\partial}{\partial Z}\right)\right\} \epsilon^2 S. \tag{12}$$

Expanding in powers of ϵ^2, it is found that the dominant dispersive terms are

$$\epsilon^8\left(S_T+\left.\frac{\partial\omega}{\partial l}\right|_0 S_X-\frac{1}{6}\left.\frac{\partial^3\omega}{\partial m^3}\right|_0 S_{ZZZ}\right)+\mathrm{i}\epsilon^{10}\left(\frac{1}{24}\left.\frac{\partial^4\omega}{\partial m^4}\right|_0 S_{ZZZZ}-\left.\frac{\partial^2\omega}{\partial l\,\partial m}\right|_0 S_{XZ}\right)+\ldots, \tag{13}$$

where use has been made of

$$\left.\frac{\partial\omega}{\partial m}\right|_0=\left.\frac{\partial^2\omega}{\partial m^2}\right|_0=0; \tag{14}$$

the above conditions express the well-known fact that the cusp line is a caustic of the linear theory.

The nonlinear and forcing terms of the evolution equation can now be obtained by a perturbation analysis that does not involve any dependence on the spatial variables X, Y, Z. Direct substitution of the expansion (7) into the free-surface conditions (2), (3) gives for the second harmonic

$$S_2=k_0 S^2, \tag{15}$$

and for the primary harmonic (correct to $O(\epsilon^6)$)

$$k_0 A+\mathrm{i}\left(\frac{2}{3}\right)^{\frac{1}{2}}k_0 S-\epsilon^6 S_T-\mathrm{i}\frac{5}{2}\left(\frac{2}{3}\right)^{\frac{1}{2}}k_0^3\,\epsilon^6 S^2S^*=0, \tag{16a}$$

$$S-\mathrm{i}\left(\frac{2}{3}\right)^{\frac{1}{2}}k_0 A+\epsilon^6 A_T+k_0^3\,\epsilon^6 S^2 S^*=-\epsilon^{-2}p\,\mathrm{e}^{-\mathrm{i}\alpha}. \tag{16b}$$

Eliminating A from the above equations, it is found that (to the same order of approximation)

$$S_T+\mathrm{i}2\left(\frac{2}{3}\right)^{\frac{1}{2}}k_0^3\,S^2S^*=-\tfrac{1}{2}\mathrm{i}\left(\frac{3}{2}\right)^{\frac{1}{2}}\epsilon^{-8}p\,\mathrm{e}^{-\mathrm{i}\alpha}. \tag{17}$$

Combining (13) with (17), it is deduced that the complete evolution equation, which includes the dominant dispersive and nonlinear effects, is

$$S_T+c_0 S_X+c_1 S_{ZZZ}+\mathrm{i}\beta S^2S^*=-\mathrm{i}\left(\frac{3}{8}\right)^{\frac{1}{2}}\epsilon^{-8}p\,\mathrm{e}^{-\mathrm{i}\alpha}, \tag{18}$$

where

$$c_0=\left.\frac{\partial\omega}{\partial l}\right|_0=\frac{1}{2^{\frac{1}{2}}},\quad c_1=-\frac{1}{6}\left.\frac{\partial^3\omega}{\partial m^3}\right|_0=\frac{2}{81},\quad \beta=2\left(\frac{3}{2}\right)^{\frac{5}{2}}. \tag{19}$$

The forcing term $p\,\mathrm{e}^{-\mathrm{i}\alpha}$ may be rewritten in terms of the envelope scales:

$$p(x,z)\,\mathrm{e}^{-\mathrm{i}\alpha}=p\left(\frac{X}{\epsilon^6},\frac{Z}{\epsilon^2}\right)\exp\left\{\mathrm{i}k_0\left[\frac{X}{\epsilon^6}\cos\gamma-\frac{Z}{\epsilon^2}\sin\gamma\right]\right\}, \tag{20}$$

so that, for fixed X, Z and $\epsilon\to 0$,

$$p\,\mathrm{e}^{-\mathrm{i}\alpha}\sim 4\pi^2\epsilon^8\hat{p}_0\,\delta(X)\,\delta(Z)\quad(\epsilon\to 0), \tag{21}$$

where $\hat{p}_0$ denotes the Fourier transform of $p(x,z)$ evaluated at $\boldsymbol{k}_0$ and δ is the Dirac delta function. Thus, combining (18) with (21), the evolution equation reads

$$S_T+c_0 S_X+c_1 S_{ZZZ}+\mathrm{i}\beta S^2S^*=\mathrm{i}c_2\,\delta(X)\,\delta(Z), \tag{22}$$

with

$$c_2=-2\pi^2\left(\frac{3}{2}\right)^{\frac{1}{2}}\hat{p}_0. \tag{23}$$

Finally, initial conditions are needed for the solution of (22); assuming that the source is turned on impulsively at $T = 0$, we require

$$S(X, Z, T) = 0 \quad (T \to 0^-). \tag{24}$$

The appearance of a delta function on the right-hand side of (22) represents the effect of the pressure source which, when viewed from the far field, has collapsed to a point. It is possible to have misgivings about this source term, since the expansion (7) is valid in the far field ($x \gg 1$), where the localized pressure distribution $p(x, z)$ vanishes. However, as will be argued in §§3 and 4 the delta function provides the correct link between the near and the far field.

3. Linear unsteady response

It is instructive to investigate first the solution, $S_L(X, Z, T)$, of the linearized version of the evolution equation (22), subject to the initial condition (24). Standard Fourier-integral methods show that

$$S_L = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}l\,\mathrm{d}m\, F(l, m; T)\exp[\mathrm{i}(lX + mZ)], \tag{25}$$

where

$$F = \frac{c_2}{4\pi^2(lc_0 - m^3c_1)}\{1 - \exp[-\mathrm{i}(lc_0 - m^3c_1)\,T]\}. \tag{26}$$

Therefore, separating the two terms in (26),

$$S_L = \frac{c_2}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}m\,\mathrm{e}^{\mathrm{i}mZ}(G_1 - G_2), \tag{27}$$

with

$$G_1(m; X) = \int_{C^+} \mathrm{d}l\,\frac{\mathrm{e}^{\mathrm{i}lX}}{lc_0 - m^3c_1}, \tag{28a}$$

$$G_2(m; X, T) = \int_{C^+} \mathrm{d}l\,\frac{\mathrm{e}^{\mathrm{i}lX}}{lc_0 - m^3c_1}\exp[-\mathrm{i}(lc_0 - m^3c_1)\,T], \tag{28b}$$

and the contour of integration C^+ is indented to pass above the pole of the integrand. Using the residue theorem, it is found that, for $X > 0$,

$$G_1 = 0 \quad (X > 0) \tag{29a}$$

and

$$G_2 = \begin{cases} -\dfrac{2\pi\mathrm{i}}{c_0}\exp(\mathrm{i}m^3c_1X/c_0) & (0 < X < c_0T) \\ & (X > c_0T). \end{cases} \tag{29b}$$

Similarly, for $X < 0$, indenting the contour of integration to pass below the pole of the integrand in (28), it is readily found that

$$G_1 = G_2 = 0 \quad (X < 0). \tag{30}$$

Therefore, combining (27), (29) and (30), it follows that

$$S_L = \frac{\mathrm{i}c_2}{2\pi c_0}\int_{-\infty}^{\infty} \mathrm{d}m\,\exp\left[\mathrm{i}\left(mZ + \frac{m^3c_1X}{c_0}\right)\right] = \frac{3\mathrm{i}c_2}{X^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{3}{2^{\frac{1}{3}}}\frac{Z}{X^{\frac{1}{3}}}\right) \quad (0 < X < c_0T) \tag{31a}$$

$$S_L = 0 \quad (X < 0, \quad X > c_0T). \tag{31b}$$

Equations (31) show that the linear response is solely confined in a finite region behind the source: for $T > 0$ and $0 < X < c_0T$, the wave disturbance is steady; for all other X, the response is zero. Close to the discontinuity at $X = c_0 T$ higher-order dispersive effects become important and give rise to a smooth transition region where the response is unsteady. This region will be discussed in detail in §6.

It is important to note that, according to (31), the linear steady state (5), derived by Ursell (1960) on the basis of the exact linear solution, is recovered as $T \to \infty$. This points to the fact that the delta function on the right-hand side of (22) indeed provides the correct matching between the near field ($x = O(1)$) and the far field ($x \gg 1$). This is so even when the nonlinear term is included in (22); as already remarked in §2, for $0 < \epsilon \ll 1$, there is an intermediate region ($X \ll 1, 1 \ll x \ll \epsilon^{-6}$) where nonlinear effects are negligible and $S \sim S_{\mathrm{L}}$. [Strictly speaking, (22) is not valid in an $O(\epsilon^6)$ region around $X = 0$, where the response is not slowly varying and the two cusp lines meet; to the leading-order approximation used here, the size of this region has shrunk to zero.]

4. Nonlinear response

One of the questions raised earlier was whether the nonlinear response reaches steady state. The regular perturbation expansion of Newman (1971) in conjunction with Lighthill's radiation condition suggest the presence of a nonlinear instability; so it is of immediate interest to examine the predictions of the uniformly valid theory developed in §2.

Returning to the nonlinear equation (22) subject to the initial condition (24), we claim that the appropriate solution satisfies

$$c_0 S_X + c_1 S_{ZZZ} + \mathrm{i}\beta S^2 S^* = 0 \quad (0 < X < c_0 T), \tag{32a}$$

with

$$S \sim S_{\mathrm{L}}(X, Z) \quad (X \to 0^+, T), \tag{32b}$$

and

$$S = 0 \quad (X < 0, \quad X > c_0 T). \tag{33}$$

This can be formally justified by adopting a frame of reference moving with c_0,

$$X' = X - c_0 T, \tag{34}$$

so that (22) reads

$$S_T + c_1 S_{ZZZ} + \mathrm{i}\beta S^2 S^* = \mathrm{i}c_2\, \delta(X' + c_0 T)\, \delta(Z) \tag{35a}$$

and

$$S(X'; Z, T) = 0 \quad (T \to 0^-). \tag{35b}$$

With this change of variable, X' can now be regarded as a parameter that denotes the position at $T = 0$ of an observer moving with speed c_0. For $T > 0$ and $X' > 0$ (i.e. $X > c_0 T$) or $T > 0$ and $X' < 0$ with $X' + c_0 T < 0$ (i.e. $X < 0$), there is no excitation on the right-hand side of (35*a*) and causality requires that the response vanishes, in agreement with (33). On the other hand, for $T > 0$ and $X' < 0$ with $X' + c_0 T > 0$ (i.e. $0 < X < c_0 T$), there is a finite response that depends on $X' + c_0 T = X$ and Z only and, therefore, has to be steady. Finally, as $X' + c_0 T = X \to 0^+$, $S \sim S_{\mathrm{L}}$ because, as already indicated, in the intermediate region $X \ll 1, 1 \ll x \ll \epsilon^{-6}$, the linear dispersive terms (of $O(X^{-\frac{4}{3}})$) dominate the nonlinear term (of $O(X^{-1})$) in (35*a*). Thus, by imposing the condition (32*b*), matching between the near and the far field is achieved and the validity of (32*a*, *b*) is verified.

Equations (32), (33) indicate that the nonlinear solution is similar in nature to the linear one; it is non-zero only in the region $0 < X < c_0 T$ and is steady. In the limit $T \to \infty$, a nonlinear steady state is approached which is governed by (32); no sign of the nonlinear instability suggested by Newman (1971) is found.

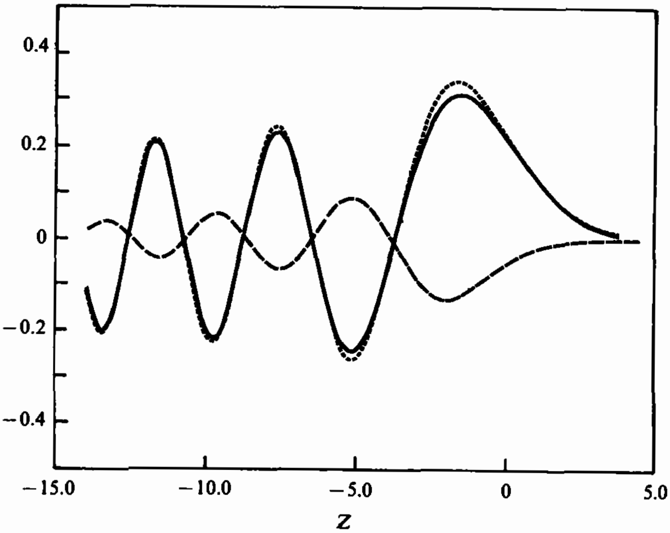

FIGURE 1. Nonlinear wave envelope: ——, real part; ----, imaginary part; ······, linear solution.

5. Numerical results

Having demonstrated that a nonlinear steady state is approached, the question arises as to how this steady state differs from the linear one. This question is addressed here by solving numerically (32*a*) subject to the asymptotic boundary condition (32*b*).

With no loss of generality, we solve (32*a*) in the normalized form

$$S_X+\tfrac{1}{3}S_{ZZZ}+\mathrm{i}S^2S=0 \quad (X>0,\quad -\infty<Z<\infty), \tag{36a}$$

with

$$S\sim\frac{1}{X^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{Z}{X^{\frac{1}{3}}}\right) \qquad (X\rightarrow 0^+,\quad -\infty<Z<\infty). \tag{36b}$$

The numerical method of solution is a conditionally stable, explicit finite-difference scheme of the Lax–Wendroff type, similar to the one used for the solution of the forced Korteweg–de Vries equation (Akylas 1984*a*). Using the notation

$$S_j^n=S(n\Delta X, j\Delta Z), \tag{37}$$

we write

$$S_j^{n+1}=S_j^n+\Delta X S_{Xj}^n+\tfrac{1}{2}\Delta X^2 S_{XXj}^n+O(\Delta X^3), \tag{38}$$

where S_X and S_{XX} are expressed in terms of S and its derivatives with respect to Z, through (36*a*); the discretization is completed by approximating all Z-derivatives in terms of second-order centred differences. Furthermore, a sufficiently large computational domain in the Z-direction was chosen so that, at the boundaries, dispersion dominated nonlinearity and S behaved in accordance with the asymptotics of the linearized solution (36*b*):

$$S\sim\begin{cases}\dfrac{1}{2\pi^{\frac{1}{2}}}(XZ)^{-\frac{1}{4}}\exp\left[-\dfrac{2}{3}\dfrac{Z^{\frac{3}{2}}}{X^{\frac{1}{2}}}\right] & (Z\gg X^{\frac{1}{3}}),\\[2ex] \dfrac{1}{\pi^{\frac{1}{2}}}(-XZ)^{-\frac{1}{4}}\sin\left[\dfrac{2}{3}\dfrac{(-Z)^{\frac{3}{2}}}{X^{\frac{1}{2}}}+\tfrac{1}{4}\pi\right] & (Z\ll -X^{\frac{1}{3}}).\end{cases} \tag{39}$$

In implementing this procedure, the solution was started at $X = 0.1$ with the linear solution (36*b*) and was advanced in X using the step sizes $\Delta Z = 0.07$ (for $-25 < Z < 5$), $\Delta X = 0.5 \times 10^{-3}$, without encountering any difficulty due to numerical instability.

Figure 1 shows the computed nonlinear complex wave envelope as a function of Z at $X = 4.0$ together with the corresponding linear solution which, according to (36*b*), was chosen to be real. The main effect of nonlinearity is to generate a non-zero imaginary part for S and, thus, give rise to a modification of the carrier wavenumber $\boldsymbol{k}_0$; the wave-amplitude distribution remains practically unaffected, for $|S|$ is found to be indistinguishable from the linear solution to within the accuracy of the graph. Exactly the same conclusions were reached at values of X as large as 15.0.

The absence of dramatic nonlinear effects in the nonlinear steady state is perhaps hinted by the fact that (36*a*) admits solitary-wave solutions only under very special circumstances (Jang & Benney 1981). Thus, one would suspect that the nonlinear response disperses out, in a way more or less similar to the linear response. However, the agreement between linear and nonlinear wave amplitudes found here is quite remarkable. From a certain point of view, this may be disappointing, but comparison with observation does not seem to contradict our findings: Newman (1970) presents aerial photographs of ship waves from which it is clear that the wave disturbance close to the cusp lines is in good agreement with linear theory; the only disagreement involves the apex of the cusp lines which is found about one ship length ahead of the ship bow. The nature of the nonlinear steady-state response found here supports the view that this discrepancy is not due to weakly nonlinear free-surface effects in the far field, but most likely should be attributed to a near-field phenomenon.

6. Unsteady transition region

The unsteady wave disturbance discussed in §§3 and 4 was found to be discontinuous at $X = c_0 T$ and it was remarked that the discontinuity is smoothed out by higher-order dispersive effects. This topic is taken up here.

The appearance of a discontinuity at $X = c_0 T$ suggests that derivatives with respect to X become large there; thus, in the vicinity of $X = c_0 T$ some higher-order dispersive terms, which are negligible away from $X = c_0 T$, become as important as the dominant dispersive and nonlinear terms retained in the evolution equation (22). For this reason, returning to (13), we note that the term proportional to S_{XZ} should be restored in (22), which now reads

$$S_T + c_0 S_X + c_1 S_{ZZZ} - \mathrm{i}\epsilon^2 c_3 S_{XZ} + \mathrm{i}\beta S^2 S^* = \mathrm{i}c_2\, \delta(X)\, \delta(Z), \tag{40}$$

where

$$c_3 = \left.\frac{\partial^2 \omega}{\partial l\, \partial m}\right|_0 = 3^{-\frac{3}{2}}. \tag{41}$$

More formally, (22) can be viewed as the outer limit of (40). The inner limit is described by $\tilde{X} = O(1)$, $\epsilon \rightarrow 0$, where $\tilde{X}$ is the inner variable:

$$\tilde{X} = \frac{X - c_0 T}{\epsilon^2}. \tag{42}$$

In terms of the inner variable, the governing equation for the inner solution $\tilde{S}(\tilde{X}, Z, T)$ is

$$\tilde{S}_T + c_1 \tilde{S}_{ZZZ} - \mathrm{i}c_3 \tilde{S}_{\tilde{X}Z} + \mathrm{i}\beta \tilde{S}^2 \tilde{S}^* = \mathrm{i}\frac{c_2}{c_0}\delta(T)\, \delta(Z), \tag{43a}$$

with the initial condition

$$\tilde{S}(\tilde{X}, Z, T) = 0 \quad (T \rightarrow 0^-). \tag{43b}$$

For $0 < T \ll 1$, when the linear dispersive terms in (43*a*) dominate the nonlinear term, $\tilde{S}$ approaches the linear similarity solution, $\tilde{S}_L$:

$$\tilde{S}_L = \frac{ic_2}{2\pi c_0} \frac{1}{T^{\frac{1}{3}}} \int_{\xi/c_3}^{\infty} ds \exp\left[i(s\zeta + c_1 s^3)\right], \tag{44}$$

where

$$\xi = \frac{\tilde{X}}{T^{\frac{2}{3}}}, \quad \zeta = \frac{Z}{T^{\frac{1}{3}}}. \tag{45}$$

Note that the outer limit of $\tilde{S}_L$ matches smoothly with the inner limit of S_L:

$$\tilde{S}_L \to 0 \quad (\tilde{X} \to \infty), \tag{46a}$$

$$\tilde{S}_L \sim \frac{ic_2}{c_0(3c_1 T)^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{3\zeta}{2^{\frac{1}{3}}}\right) = S_L(X \to c_0 T^-, Z) \quad (\tilde{X} \to -\infty). \tag{46b}$$

Accordingly, the solution of the nonlinear forced problem (43) is equivalent to the solution of the homogeneous equation

$$\tilde{S}_T + c_1 \tilde{S}_{ZZZ} - ic_3 \tilde{S}_{\tilde{X}Z} + i\beta \tilde{S}^2 \tilde{S}^* = 0, \tag{47}$$

subject to the asymptotic initial condition

$$\tilde{S} \to \tilde{S}_L \quad (T \to 0^+), \tag{48}$$

and the matching requirements

$$\tilde{S} \to 0 \quad (\tilde{X} \to +\infty), \quad \tilde{S} \to S(X \to c_0 T^-, Z) \quad (\tilde{X} \to -\infty). \tag{49}$$

The above analysis shows that all unsteady effects are confined in the region $X - c_0 T = O(\epsilon^2)$ which, as T increases, moves away from the source with speed c_0. Also, from (44), it is easy to show that, for $\tilde{X} = O(1)$ and $Z \to +\infty$, the unsteady response decays only algebraically rather than exponentially; thus, it is possible for the unsteady disturbance to persist outside the region where the steady response is appreciable. More detailed information about the effect of nonlinearity inside the unsteady region requires a numerical investigation of the nonlinear problem (47)–(49) and is not pursued here.

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